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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1153

CALCULATION OF THE PRESSURE DISTRIBUTION ON BODIES OF REVOLUTION IN THE SUBSONIC FLOW OF A GAS

PART I - AXIALLY SYMMETRICAL FLOW

By Herbert Bilharz and Ernst Hölder

Translation

"Zur Berechnung der Druckverteilung an Rotationskörpern in der Unterschallströmung eines Gases Teil I: Achsensymmetrische Strömung"

Deutsche Luftfahrtforschung, Forschungsbericht Nr. 1169/1

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I. INTRODUCTION

The present report concerns a method of computing the velocity and pressure distributions on bodies of revolution in axially symmetrical flow in the subsonic range.

The differential equation for the velocity potential Φ of a compressible fluid motion is linearized in the conventional manner, and then put in the form $\bar{\Delta}\bar{\Phi} = 0$ by affine transformation. The quantity $\bar{\Phi}$ represents the velocity potential of a fictitious incompressible flow, for which a constant superposition of sources by sections is secured by a method patterned after von Kármán (reference 1) which must comply with the boundary condition $\frac{\partial\Phi}{\partial n} = 0$ at the originally specified contour. This requirement yields for the "pseudo-stream function" $\bar{\Psi}$ a differential equation which must be fulfilled for as many points on the contour as source lengths are assumed. In this manner, the problem of defining the still unknown source intensities is reduced to the solution of an inhomogeneous equation system. The pressure distribution is then determined with the aid of Bernoulli's equation and the adiabatic equation of state.

Lastly, the pressure distributions in compressible and incompressible medium are compared on a model problem.

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*"Zur Berechnung der Druckverteilung an Rotationskörpern in der Unterschallströmung eines Gases Teil I: Achsensymmetrische Strömung," Zentrale für wissenschaftliches Berichtswesen über Luftfahrtforschung (ZWB) Berlin-Adlershof, Forschungsbericht Nr. 1169/1, Braunschweig, Jan. 15, 1940.

II. STATIONARY POTENTIAL FLOW

The discussion is restricted to a frictionless compressible fluid and it is assumed that the density of the medium is solely dependent upon the pressure p, while the functional relation between pressure and density is to be monotonic and continously differentiable.

The steady motion follows then, for external conservative forces, the law of Lagrange analogous to the frictionless homogeneous incompressible fluid: If the frictionless compressible fluid moves irrotationally at time interval t_0, then it does so in every subsequent time interval $t > t_0$. Since the state of rest is a special case of the irrotational motion, the flow of a compressible fluid (nonstationary at first) starting from rest is free from rotation and certainly remains so as long as the flow velocity at no point becomes greater than the velocity of sound of the medium. An irrotational motion, on the other hand, can always be represented by a velocity potential Φ, so that the velocity vector $\underline{v}$[1] becomes the gradient of this potential.

$$\underline{v} = \operatorname{grad} \Phi \tag{1}$$

with x, y, z denoting the rectangular Cartesian coordinates of the space (γ) in which the fluid moves, and u, v, w the components of $\underline{v}$ in the directions of the axes of coordinates. Equation (1), expressed by components, reads

$$u = \frac{\partial \Phi}{\partial x},\ v = \frac{\partial \Phi}{\partial y},\ w = \frac{\partial \Phi}{\partial z} \tag{2}$$

On the assumption

$$\rho = f(p) \tag{3}$$

the equation of motion of the compressible medium can be formally transformed into that of the incompressible medium by replacing in the latter, $\frac{p}{\rho}$ by the pressure function

$$P(\rho) = \int_{p_0}^{p} \frac{dp}{\rho} \tag{4}$$

[1]$\underline{v}$ is used throughout the test and figures of this report in place of $\mathfrak{w}$, which was used in the original German report.

In the absence of external forces and in stationary flow, as invariably assumed in the following, Bernoulli's equation reads

$$\frac{\underline{v}^2}{2} + P(\rho) = \text{const.} \tag{5}$$

Lastly, the equation of continuity is needed:

$$\frac{1}{\rho}\,\text{div}\,(\rho\underline{v}) = \text{div}\,\underline{v} + \frac{1}{\rho}\,\underline{v} \circ \text{grad}\,\rho = 0 \tag{6}$$

The flow is then defined by (1), (5), (6), and the subsequently discussed boundary condition (21).

Now

$$\frac{1}{\rho}\,\text{grad}\,\rho = \frac{1}{\rho}\frac{d\rho}{dp}\,\text{grad}\,p = \frac{d\rho}{dp}\,\text{grad}\,P$$

hence by (6) in conjunction with (5)

$$\text{div}\,\underline{v} - \frac{d\rho}{dp}\,\underline{v} \circ \text{grad}\,\frac{\underline{v}^2}{2} = 0 \tag{7}$$

Owing to

$$\begin{aligned}\underline{v} \circ \text{grad}\,\frac{\underline{v}^2}{2} &= \underline{v} \circ \nabla\,\underline{v} \circ \underline{v} \\ &= \breve{u}^2\frac{\partial \breve{u}}{\partial x} + v^2\frac{\partial v}{\partial y} + w^2\frac{\partial w}{\partial z} + \breve{u}w\left(\frac{\partial \breve{u}}{\partial y} + \frac{\partial v}{\partial x}\right) \\ &\quad + vw\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + w\breve{u}\left(\frac{\partial w}{\partial x} + \frac{\partial \breve{u}}{\partial z}\right)\end{aligned}$$

equation (7) becomes, with observation of irrotationality, in coordinates,

$$\frac{\partial \breve{u}}{\partial x}\left(1 - \frac{\breve{u}^2}{c^2}\right) + \frac{\partial v}{\partial y}\left(1 - \frac{v^2}{c^2}\right) + \frac{\partial w}{\partial z}\left(1 - \frac{w^2}{c^2}\right) - \frac{2}{c^2}\left(\breve{u}v \frac{\partial \breve{u}}{\partial y} + vw \frac{\partial v}{\partial z} + w\breve{u} \frac{\partial w}{\partial x}\right) = 0 \tag{8}$$

with

$$c = +\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}$$

the local velocity of sound.

If the adiabatic equation of state

$$p = p_\infty \left(\frac{\rho}{\rho_\infty}\right)^\kappa \tag{9}$$

is used as basis, equation (3) is satisfied; quantity p represents the pressure at any point of the medium and p_∞ the pressure in the (undisturbed) flow at infinity; ρ and ρ_∞ are the corresponding densities, $\kappa = \frac{c_p}{c_v}$ is the ratio of specific heat of the gas (approximately 1.41 for air). By (5) and (9) the local sonic velocity at any point of the medium with the existing velocity is

$$c^2 = \kappa \frac{p_\infty}{\rho_\infty} + \frac{\kappa - 1}{2}\left(\underline{v}_\infty^2 - \underline{v}^2\right) \tag{10}$$

and

$$c^2 = c_\infty^2 \left[1 + \frac{\kappa - 1}{2}\left(1 - \frac{\underline{v}^2}{\underline{v}_\infty^2}\right)\frac{\underline{v}_\infty^2}{c_\infty^2}\right] \tag{10*}$$

Therefore, c has a maximum value for $\underline{v} = 0$, that is, at the stagnation point, where

$$c_{max}^2 = c_\infty^2 \left(1 + \frac{\kappa - 1}{2}\frac{\underline{v}_\infty^2}{c_\infty^2}\right)$$

The critical velocity also follows from (10*) when $c = |\underline{v}|$ and resolved with respect to $\underline{v}$:

$$v_{crit}^2 = c_{min}^2 = \frac{2}{\kappa + 1} c_\infty^2 \left(1 + \frac{\kappa - 1}{2} \frac{v_\infty^2}{c_\infty^2}\right)$$

for

$$\frac{|v_\infty|}{c_\infty} = 0.8 \quad \text{and} \quad \kappa = 1.408$$

for instance

$$c_{max} = 1.0633 c_\infty \quad \text{and} \quad c_{min} = 0.96902 c_\infty$$

Since c_{max} is assumed at the contour while elsewhere in the medium

$$c_{min} < c \leq c_{max}$$

c may be put equal to c_∞ in first approximation, by reason of

$$\frac{c_{max} - c_{min}}{c} = 0.0887 \left(1 + 0.204 \frac{v_\infty^2}{c_\infty^2}\right)^{\frac{1}{2}} \leq 0.0973$$

particularly as $c \longrightarrow c_\infty$ with increasing distance from the contour as a result of the continuity of the flow process.

III. LINEARIZATION

From (8) it is seen that in all cases where all velocity components are negligibly small compared to the sonic velocity c_{min}, equation (8) leads to equation

$$\operatorname{div} \underline{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \Delta \Phi = 0 \tag{11}$$

If the resulting velocity is no longer sufficiently small but still below c_{min}, it is to be expected that the quantitative relations in the compressible medium are altered, but the character of the flow remains similar to the incompressible medium. It is always assumed in the following that $\underline{v}^2 < c_{min}^2$ in the entire range of flow.

With (2), equation (8) is written in the form

$$\left(1 - \frac{\breve{u}^2}{c^2}\right)\frac{\partial^2\Phi}{\partial x^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2\Phi}{\partial y^2}$$

$$+ \left(1 - \frac{w^2}{c^2}\right)\frac{\partial^2\Phi}{\partial z^2} - \frac{2}{c^2}\left(\breve{u}v\,\frac{\partial^2\Phi}{\partial x\partial y} + vw\,\frac{\partial^2\Phi}{\partial y\partial z} + w\breve{u}\,\frac{\partial^2\Phi}{\partial x\partial z}\right) = 0 \tag{12}$$

and the values at infinity substituted in the coefficients of the second derivatives of Φ for u, v, w and c. Hence there follows by (12) a linear partial differential equation of the second order of the elliptical type with constant coefficients.

If the x-axis of the coordinate system γ points in direction of the velocity at infinity and

$$0 < \lambda = \frac{|\underline{v}_\infty|}{c_\infty} = \frac{\breve{u}_\infty}{c_\infty} < 1$$

denotes the ratio of undisturbed stream velocity to sonic velocity at infinity, the substitution

$$T:\quad \breve{u} = \breve{u}_\infty,\quad v = 0,\quad w = 0;\ c = c_\infty$$

gives an appropriate linearization for (12).

For the velocity potential Φ the equation reads

$$(1 - \lambda^2)\,\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{13}$$

It should be borne in mind that the linearization loses its validity near the stagnation points.

IV. PSEUDO-STREAM FUNCTION IN (γ)

Let

$$\Lambda = +(1 - \lambda^2)^{-\frac{1}{2}}$$

By affine transformation

$$A: \quad \bar{x} = x, \ \bar{y} = y\Lambda^{-1}, \ \bar{z} = z\Lambda^{-1}$$

the system changes to $\bar{\gamma}$. The equation

$$A\Phi\ (x, y, z) = \bar{\Phi}(\bar{x}, \bar{y}, \bar{z})$$

is to imply that to the points $P(x, y, z)$ and $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ coordinated by A equal potential values are to correspond (transplanting of potential Φ from space (γ) in space $(\bar{\gamma})$).

Owing to

$$\frac{\partial^2\bar{\Phi}}{\partial\bar{x}^2} = \frac{\partial^2\Phi}{\partial x^2}; \ \frac{\partial^2\bar{\Phi}}{\partial\bar{y}^2} = \Lambda^2\frac{\partial^2\Phi}{\partial y^2}; \ \frac{\partial^2\bar{\Phi}}{\partial\bar{z}^2} = \Lambda^2\frac{\partial^2\Phi}{\partial z^2}$$

and (13), the potential $\bar{\Phi}$ therefore satisfies the Laplace equation

$$\bar{\Delta}\bar{\Phi} \equiv \frac{\partial^2\bar{\Phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\Phi}}{\partial\bar{y}^2} + \frac{\partial^2\bar{\Phi}}{\partial\bar{z}^2} = 0 \tag{14}$$

The transformation A therefore yields the rule: The flow around a contour $\underline{C}$[2] in the compressible medium of a space (γ) can be computed in linearized form in an incompressible medium of a space $(\bar{\gamma})$ by reason of the fact that in $(\bar{\gamma})$ the contour

$$\bar{\underline{C}} = A\underline{C}$$

affinely conjugated to $\underline{C}$ is used as basis.

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[2] $\underline{C}$ is used throughout the text and figures of this report in place of the symbol $\mathcal{C}$ used in the orginal German report.

The following is restricted to axially symmetrical bodies and axially symmetrical flow; the x-axis is the axis of rotation.

Putting

$$\overline{\varphi}(\bar{x}, \bar{y}) = \overline{\Phi}(\bar{x}, \bar{y}, 0) \tag{15}$$

we get, since the local function $\overline{\Phi}$ on each circle orthogonal to the x-axis has a constant value,

$$\overline{\varphi}(\bar{x}, \bar{r}) = \overline{\Phi}(\bar{x}, \bar{y}, \bar{z}) = \overline{\Phi}(\bar{x}, \bar{r}, 0)$$

with

$$\bar{r} = +(\bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}$$

and after some conversions

$$\overline{\Delta\Phi} = \frac{\partial^2\bar{\varphi}}{\partial \bar{x}^2} + \frac{1}{r}\frac{\partial\varphi}{\partial \bar{r}} + \frac{\partial^2\bar{\varphi}}{\partial \bar{r}^2} = 0, \qquad (\bar{r} > 0) \tag{16}$$

as the differential equation for the function $\overline{\varphi}$.

For this potential $\overline{\varphi}$ there exists a function $\overline{\Psi}$, continuous with its first and second partial derivatives and well defined up to an additive constant (not the Stokes stream function) which satisfies the equation

$$\frac{\partial\overline{\Psi}}{\partial \bar{x}} = -\bar{r}\frac{\partial\overline{\varphi}}{\partial \bar{r}}, \quad \frac{\partial\overline{\Psi}}{\partial \bar{r}} = \bar{r}\frac{\partial\overline{\varphi}}{\partial \bar{x}} \tag{17}$$

The integrability condition for (17) reads

$$\frac{\partial}{\partial \bar{r}}\left(-\bar{r}\frac{\partial\overline{\varphi}}{\partial \bar{r}}\right) = \frac{\partial}{\partial \bar{x}}\left(\bar{r}\frac{\partial\overline{\varphi}}{\partial \bar{x}}\right) \tag{18}$$

or computed

$$\bar{r}\frac{\partial^2\bar{\varphi}}{\partial\bar{r}^2} + \frac{\partial\bar{\varphi}}{\partial\bar{r}} + \bar{r}\frac{\partial^2\bar{\varphi}}{\partial\bar{x}^2} = 0$$

therefore (18) is fulfilled because of (16). By (17) then

$$\bar{\Psi}(\bar{x}, \bar{r}) = \int_{(x_0,\bar{r}_0)}^{(\bar{x},\bar{r})} \bar{r}\left(\frac{\partial\bar{\varphi}}{\partial\bar{x}}d\bar{r} - \frac{\partial\bar{\varphi}}{\partial\bar{r}}d\bar{x}\right)$$

By $(\bar{x}_0, \bar{r}_0)$ is meant any point in the definition range of the function $\bar{\varphi}(\bar{x}, \bar{r})$, that is, any point of regularity of $\bar{\Phi}(\bar{x}, \bar{y}, \bar{z})$ located on the plane $\bar{z} = 0$.

By (15) and (17) the velocity component of $\bar{\underline{v}}$ in axial and radial direction in a meridian plane (such as $\bar{z} = 0$, for instance) is

$$\left.\begin{aligned}\bar{u} &= \bar{u}_{\bar{x}} = \frac{\partial\bar{\Phi}}{\partial\bar{x}} = \frac{\partial\bar{\varphi}}{\partial\bar{x}} = \frac{1}{\bar{r}}\frac{\partial\bar{\Psi}}{\partial\bar{r}}\\ \bar{v} &= \bar{u}_{\bar{r}} = \frac{\partial\bar{\Phi}}{\partial\bar{r}} = \frac{\partial\bar{\varphi}}{\partial\bar{r}} = -\frac{1}{\bar{r}}\frac{\partial\bar{\Psi}}{\partial\bar{x}}\end{aligned}\right\} \quad (\bar{r} > 0) \tag{19}$$

Introducing the terms resulting from (19) into the relation

$$\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{\varphi}}{\partial\bar{r}}\right) = \frac{\partial}{\partial\bar{r}}\left(\frac{\partial\bar{\varphi}}{\partial\bar{x}}\right)$$

finally gives the pseudo-stream function $\bar{\Psi}$ in $(\bar{\gamma})$:

$$\frac{\partial^2\bar{\Psi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\Psi}}{\partial\bar{r}^2} - \frac{1}{r}\frac{\partial\bar{\Psi}}{\partial\bar{r}} = 0 \tag{20}$$

V. BOUNDARY CONDITION FOR $\overline{\Psi}$

Since $\underline{C}$ in (γ) is streamline, the boundary condition

$$\frac{\partial \Phi}{\partial n} = 0 \tag{21}$$

on $\underline{C}$ must be fulfilled for Φ.

The derivation of the condition resulting from (21) for $\overline{\Psi}$ in in $(\overline{\gamma})$ gives

$$\frac{\partial \Phi}{\partial n} = \frac{\partial \Phi}{\partial x} \cos (n, x) + \frac{\partial \Phi}{\partial r} \cos (n, r) = 0$$

by (21).

Since

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \overline{\Phi}}{\partial \bar{x}} = \frac{1}{\bar{r}} \frac{\partial \overline{\Psi}}{\partial \bar{r}}$$

$$\frac{\partial \Phi}{\partial r} = \frac{1}{\Lambda} \frac{\partial \overline{\Phi}}{\partial \bar{r}} = - \frac{1}{\Lambda \bar{r}} \frac{\partial \overline{\Psi}}{\partial \bar{x}}$$

according to A and (17) and since

$$\frac{dr}{dx} = \frac{dr}{d\bar{r}} \frac{d\bar{r}}{d\bar{x}} = \Lambda \frac{d\bar{r}}{d\bar{x}} = - \operatorname{ctg} (n, x)$$

we get

$$- \frac{\Lambda}{\bar{r}} \frac{\partial \overline{\Psi}}{\partial \bar{r}} \frac{d\bar{r}}{d\bar{x}} - \frac{1}{\Lambda \bar{r}} \frac{\partial \overline{\Psi}}{\partial \bar{x}} = 0$$

hence

$$\Lambda^2 \frac{\partial\bar{\psi}}{\partial\bar{r}} \frac{d\bar{r}}{d\bar{x}} + \frac{\partial\bar{\psi}}{\partial\bar{x}} = 0 \tag{22}$$

or

$$\Lambda \frac{\partial\bar{\psi}}{\partial\bar{r}} \frac{dr}{dx} + \frac{\partial\bar{\psi}}{\partial\bar{x}} = 0$$

$\frac{dr}{dx}$ = the direction of the tangent (as in the plane $z = 0$, for instance) to the contour.

In correspondence with the flow velocity $(u_\infty, 0)$ at infinity, $\bar{\psi} = \frac{1}{2}\breve{u}_\infty\bar{r}^2 +$ by (17).

VI. CONSTANT AXIAL SUPERPOSITION BY SECTIONS

After introducing three-dimensional polar coordinates $\bar{\rho}$, $\bar{\vartheta}$, $\bar{x}$ in $(\bar{\gamma})$, that is,

$\bar{\rho}$ length of vector

$\bar{\vartheta}$ angle between vector and axis $\bar{x}$

$\bar{\chi}$ angle of meridian plane with plane $\bar{z} = 0$

the velocity components are

$$\breve{u}_{\bar{\rho}} = \frac{\partial\bar{\Phi}}{\partial\bar{\rho}}, \quad \breve{u}_{\bar{\vartheta}} = \frac{1}{\bar{\rho}}\frac{\partial\bar{\Phi}}{\partial\bar{\vartheta}}, \quad \breve{u}_{\bar{x}} = 0 \tag{23}$$

The velocity potential for a simple source of yield Q in the origin is

$$\bar{\Phi}_* = -\frac{Q}{4\pi\bar{\rho}}$$

On the basis of the spherical symmetry the stream function $\overline{\Psi}_*$ is analogous to (19)

$$\frac{\partial \overline{\Phi}_*}{\partial \bar{\rho}} = \frac{1}{\bar{\rho}^2 \sin \bar{\vartheta}} \frac{\partial \overline{\Psi}_*}{\partial \bar{\vartheta}}$$

$$\frac{1}{\rho} \frac{\partial \overline{\Phi}_*}{\partial \bar{\vartheta}} = - \frac{1}{\bar{\rho} \sin \bar{\vartheta}} \frac{\partial \overline{\psi}_*}{\partial \bar{\rho}}$$

hence

$$\overline{\psi}_* = - \frac{Q}{4\pi} (1 + \cos \bar{\vartheta})$$

The stream function $\overline{\psi}_o$ and the velocity component is now computed for a source of length a and constant yield q per unit length.

The contribution of an element $d\bar{\xi}$ to $\overline{\Psi}_o$ in point $\bar{P}$ amounts to

$$d\overline{\psi}_o = - \frac{q}{4\pi} (1 + \cos \bar{\vartheta})\, d\bar{\xi}$$

hence $\overline{\psi}_o$ is for the total length a

$$\overline{\Psi}_o = - \frac{q}{4\pi} \int_0^a (1 + \cos \bar{\vartheta})\, d\bar{\xi} = - \frac{q}{4\pi} (a + \bar{\rho}' - \bar{\rho}'') \qquad (24)$$

the latter because

$$d\bar{\xi} \cos \bar{\vartheta} = - d\bar{\rho}$$

With

$$Q = qa$$

as the total yield of a, (24) becomes

$$\bar{\psi}_o = -\frac{Q}{4\pi}\left(1 + \frac{\bar{\rho}' - \bar{\rho}''}{a}\right) \tag{25}$$

By (19) the axial and radial velocity components are

$$\left.\begin{aligned} \bar{u}_{\bar{x}} &= \frac{1}{\bar{r}}\frac{\partial \bar{\psi}_o}{\partial \bar{r}} = \frac{Q}{4\pi\ a\ \bar{r}}\left(\frac{\partial \bar{\rho}''}{\partial \bar{r}} - \frac{\partial \bar{\rho}'}{\partial \bar{r}}\right) \\ \bar{u}_{\bar{r}} &= -\frac{1}{\bar{r}}\frac{\partial \bar{\psi}_o}{\partial \bar{x}} = -\frac{Q}{4\pi\ a\ \bar{r}}\left(\frac{\partial \bar{\rho}''}{\partial \bar{x}} - \frac{\partial \bar{\rho}'}{\partial \bar{x}}\right) \end{aligned}\right\} \tag{26}$$

Because

$$\bar{\rho} = +(\bar{x}^2 + \bar{r}^2)^{\frac{1}{2}}$$

we get

$$\frac{\partial \bar{\rho}}{\partial \bar{r}} = \frac{\bar{r}}{\bar{\rho}} = \sin \bar{\vartheta}$$

$$\frac{\partial \bar{\rho}}{\partial \bar{x}} = \frac{\bar{x}}{\bar{\rho}} = \cos \bar{\vartheta}$$

hence finally by (26)

$$\left.\begin{aligned} \bar{\breve{u}}_{\bar{x}} &= \frac{Q}{4\pi\, a\, \bar{r}} \left(\sin \bar{\vartheta}'' - \sin \bar{\vartheta}'\right) \\ \bar{\breve{u}}_{\bar{r}} &= - \frac{Q}{4\pi\, a\, \bar{r}} \left(\cos \bar{\vartheta}'' - \cos \bar{\vartheta}'\right) \end{aligned}\right\} \tag{27}$$

Assuming consecutive source distances of constant length a and yield Q_i $(i = 1, 2, \ldots, n)$ on the x axis (≥ 1), and denoting the distances of a particular streamline point P from the end points of the i-th source distance with $\bar{\rho}_i' = \bar{\rho}_i$ and $\bar{\rho}_i'' = \bar{\rho}_{i+1}$ $(i = 1, \ldots, n)$ we get by (25)

$$\sum_{i=1}^{n} \bar{\psi}_{oi} = - \frac{1}{4\pi} \sum_{i=1}^{n} \left(1 + \frac{\bar{\rho}_i - \bar{\rho}_{i+1}}{a}\right) Q_j \tag{28}$$

This "stream function" is then superposed on that of the incompressible parallel flow (approach in positive $\bar{x}$ direction)

$$\bar{\Psi}_1 = \breve{u}_\infty \frac{\bar{r}^2}{2} = U \frac{\bar{r}^2}{2}$$

so that the total flow is

$$\bar{\psi} = U \frac{\bar{r}^2}{2} - \sum_{i=1}^{n} \frac{Q_i}{4\pi} \left(1 + \frac{\bar{\rho}_i - \bar{\rho}_{i+1}}{a}\right) \tag{29}$$

While the curves $\bar{\psi}$ = constant in the incompressible medium in $(\bar{\gamma})$ represent streamlines, the curve $\bar{\psi} = 0$ consists of especially the $\bar{x}$-axis and $\bar{C}$, and $\bar{\psi} = 0$ must be applied for exactly as many points of $\bar{C}$ as there are constant source distances of length a with unknown intensity Q_i- which give Q_i by solution of a linear nonhomogeneous equation system. A somewhat different method must be applied in the compressible medium.

VII. DETERMINATION OF SOURCE INTENSITIES

The dimensionless quantities

$$z_i = \frac{Q_i}{2\bar{r}} Ua^2 \tag{30}$$

are introduced as unknowns; $\bar{\rho}_{i,k}' = \bar{\rho}_{i,k}$ and $\bar{\rho}_{i,k}''$ indicate the length of the vectors reaching from the end points of the i-th source distance to the k-th boundary point on $\bar{C}$. By reason of the uninterrupted sequence of the source distances

$$\bar{\rho}_{i,k}'' = \bar{\rho}_{i+1,k}' \quad (i, k, = 1, 2, \ldots, n)$$

The radius at the k-th contour point is indicated with $\bar{r}_k$; the boundary points on $\bar{C}$ are to be so assumed that $\bar{r}_k$ (k = 1, . . ., n) is median perpendicular of the k-th source distance.

Proceeding from (29) the linear equation system in the unknowns z_i is set up. By it, in conjunction with, (30)

$$\bar{\psi} = U\frac{\bar{r}^2}{2} - \sum_{i=1}^{n} \frac{z_i U a^2}{2}\left(1 + \frac{\bar{\rho}_i - \bar{\rho}_{i+1}}{a}\right) \tag{31}$$

and by (27)

$$\left.\begin{aligned}\frac{\partial\bar{\psi}}{\partial\bar{x}} &= -\sum_{i=1}^{n}\frac{z_i\,U\,a}{2}\left(\cos\bar{\vartheta}_i - \cos\bar{\vartheta}_{i+1}\right)\\ \frac{\partial\bar{\psi}}{\partial\bar{r}} &= U\bar{r} - \sum_{i=1}^{n}\frac{z_i\,U\,a}{2}\left(\sin\bar{\vartheta}_i - \sin\bar{\vartheta}_{i+1}\right)\end{aligned}\right\} \tag{32}$$

The boundary condition (22) for $\bar{\psi}$ must be fulfilled from (31). The final result with due allowance for (32) is

$$\frac{\bar{r}}{a}\,\Lambda^2\frac{d\bar{r}}{d\bar{x}} - \Lambda^2\frac{d\bar{r}}{d\bar{x}}\sum_{i=1}^{n}\frac{z_i}{2}\left(\sin\bar{\vartheta}_i - \sin\bar{\vartheta}_{i+1}\right) - \sum_{i=1}^{n}\frac{z_i}{2}\left(\cos\bar{\vartheta}_i - \cos\bar{\vartheta}_{i+1}\right) = 0 \tag{33}$$

This equation must be satisfied for n points on $\bar{C}$, which have the distances $\bar{r}_k$ $(k = 1, 2, \ldots n)$ from the x axis; (33) is therefore written in the form

$$\sum_{i=1}^{n} c_{ik}\, z_i = \frac{r_k}{a}\left(\frac{dr}{dx}\right)_k \quad (k = 1, 2, \ldots, n) \tag{34}$$

r_k and $\left(\frac{dr}{dx}\right)_k$ refer to the originally given contour C in (γ). For the coefficients c_{ik}

$$c_{ik} = \frac{1}{2}\Lambda\left(\frac{dr}{dx}\right)_k (\sin \bar{\vartheta}_{ik} - \sin \bar{\vartheta}_{i+1,k})$$

$$+ \frac{1}{2}(\cos \bar{\vartheta}_{i,k} - \cos \bar{\vartheta}_{i+1,k}) \tag{35}$$

Transforming (34) so that the new coefficients c^*_{ik} have the property

$$c^*_{kk} = 1 \quad (k = 1, 2, \ldots, n) \tag{36}$$

we get the equation system

$$\sum_{i=1}^{n} c^*_{ik} z_i = \frac{r}{a}\left(\frac{dr}{dx}\right)_k \frac{1}{\cos \bar{\vartheta}_{kk}} \quad (k = 1, \ldots, n) \tag{37}$$

with

$$c^*_{ik} = c_{ik} \frac{1}{\cos \bar{\vartheta}_{kk}}$$

where $|c_{ik}|$ is small, when $|i - k|$ is great.

Putting

$$\alpha_k = \text{arc tan } \Lambda\left(\frac{dr}{dx}\right)_k$$

finally gives

$$c^*_{ik} = \frac{1}{\cos \alpha_k \cos \bar{\vartheta}_{kk}} \sin\left(\frac{\bar{\vartheta}_{i+1,k} + \bar{\vartheta}_{i,k}}{2} - \alpha_k\right) \sin\left(\frac{\bar{\vartheta}_{i+1,k} - \bar{\vartheta}_{i,k}}{2}\right) \tag{38}$$

The formulas (36) and (38) prove themselves suitable for the numerical calculation by iteration.

VIII. DETERMINATION OF VELOCITY DISTRIBUTION

By (17)

$$\frac{\partial\overline{\psi}}{\partial\bar{x}} = -\bar{r}\frac{\partial\overline{\varphi}}{\partial\bar{r}} = -r\frac{\partial\Phi}{\partial r} \qquad \frac{\partial\overline{\psi}}{\partial\bar{r}} = \bar{r}\frac{\partial\overline{\varphi}}{\partial\bar{x}} = \bar{r}\frac{\partial\Phi}{\partial x}$$

hence the axial and radial velocity components on **C**:

$$\breve{u}_x = \frac{\partial\Phi}{\partial x} = \frac{1}{\bar{r}}\frac{\partial\overline{\psi}}{\partial\bar{r}}, \quad \breve{u}_r = \frac{\partial\Phi}{\partial r} = -\frac{1}{r}\frac{\partial\overline{\psi}}{\partial\bar{x}} \tag{39}$$

and the square of the magnitude of velocity

$$\underline{v}^2 = \breve{u}_x^2 + \breve{u}_r^2 = \frac{1}{r^2}\left(\overline{\psi}_{\bar{x}}^2 + \Lambda^2\overline{\psi}_{\bar{r}}^2\right) = \frac{1}{\Lambda^2}\left(\frac{\overline{\psi}_{\bar{x}}}{\bar{r}}\right)^2 + \left(\frac{\overline{\psi}_{\bar{r}}}{\bar{r}}\right)^2 \tag{40}$$

After computing the z_i from (37), $\overline{\psi}_{\bar{x}}$ and $\overline{\psi}_{\bar{r}}$ are known by (32), hence by (39) and (40) the velocity component.

IX. PRESSURE DISTRIBUTION

By (10*) with the aid of (5) and (9)

$$\frac{p}{p_\infty} = \left(\frac{\rho}{\rho_\infty}\right)^\kappa = \left[1 + \frac{\kappa - 1}{2}\lambda^2\left(1 - \frac{\underline{v}^2}{\underline{v}_\infty^2}\right)\right]^{\frac{\kappa}{\kappa-1}} \tag{41}$$

and from it by expansion in powers of λ^2 (terms of power higher than the first in λ^2 being disregarded): (See reference 2.)

$$\frac{p - p_\infty}{\frac{\rho}{2}\underline{v}_\infty^2} = \left(1 - \frac{\underline{v}^2}{\underline{v}_\infty^2}\right) + \frac{\lambda^2}{4}\left(1 - \frac{\underline{v}^2}{\underline{v}_\infty^2}\right)^2 \tag{42}$$

while $\underline{v}_\infty^2 = U^2$ and $\underline{v}^2$ from (40) must be inserted in (41) and (42).

Since the line of reasoning resulting in (42) holds only for small λ, a different method for the pressure distribution is indicated, which holds for all λ with

$$0 \leq \lambda = \frac{|\underline{v}_\infty|}{c_\infty} \leq \max \frac{|\underline{v}|}{c} \leq 1$$

and which gives the Mach number at any point of the body and from it the local velocity of sound c.

The quantity $\bar{\underline{w}}$ is introduced through the equation

$$\bar{\underline{w}}^2 = \frac{2\kappa}{\kappa - 1}\frac{p_\infty}{\rho_\infty} + \underline{v}_\infty^2$$

by (48) $\bar{\underline{w}}$ proves itself as maximum velocity; and (10) becomes then

$$\bar{c}^2 = \frac{\kappa - 1}{2}(1 - \underline{w}^2) \tag{43}$$

with

$$\underline{w}^2 = \frac{\underline{v}^2}{\bar{\underline{w}}^2}, \quad \bar{c}^2 = \frac{c^2}{\bar{\underline{w}}^2}$$

with c replacing $\bar{c}$, hence putting $|\bar{\underline{w}}| = 1$ the square μ of the Mach number is by (43)

$$\mu = \frac{w^2}{c^2} = \frac{2}{\kappa - 1} \frac{\underline{w}^2}{1 - \underline{w}^2} \tag{44}$$

and therefore

$$|\underline{w}| = + \left[\frac{(\kappa - 1)\mu}{2 + (\kappa - 1)\mu} \right]^{\frac{1}{2}} \tag{45}$$

From (4) and (9) follows

$$dP = \frac{dp}{\rho} = C_o \rho^{\kappa - 2} d\rho$$

and from it by integration

$$P = C_1 \rho^{\kappa - 1} = C_1 e^{(\kappa - 1) \log \rho} \tag{46}$$

$C_o = \dfrac{\kappa p_o}{\rho_o{}^{\kappa}}$ and C_1 are constant.

By (46)

$$c^2 = \frac{dp}{d\rho} = \frac{dP}{d \log \rho} = (\kappa - 1) P \tag{47}$$

On the other hand, by (5)

$$\frac{1}{2}\underline{v}^2 + P = \frac{1}{2}\bar{\underline{w}}^2 \tag{48}$$

hence, because of $\bar{\underline{w}}^2 = 1$

$$p = \frac{1}{2}(1 - \underline{w}^2)$$

Therefore by (47) in conjunction with (43)

$$\frac{\kappa - 1}{2}(1 - \underline{w}^2) = c^2 = (\kappa - 1)\ p = (\kappa - 1)\ C_1\ \rho^{\kappa-1}$$

hence

$$C_1\ \rho^{\kappa-1} = \frac{1}{2}(1 - \underline{w}^2) \tag{49}$$

with ρ_0 as the density in the stagnation point the integration constant C_1 becomes

$$C_1 = \frac{1}{2}\rho_0^{\ 1-\kappa}$$

hence by (49) finally

$$\left(\frac{\rho}{\rho_0}\right)^{\kappa-1} = 1 - \underline{w}^2 \tag{50}$$

If p_0 is the pressure in the stagnation point

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\kappa} = (1 - \underline{w}^2)^{\frac{\kappa}{\kappa-1}}$$

by (50), whence by (44)

$$\frac{p_o}{p} = \left(1 + \frac{\kappa - 1}{2}\mu\right)^{\frac{\kappa}{\kappa-1}}$$

therefore

$$\mu = \frac{2}{\kappa - 1}\left[\left(\frac{p_o}{p}\right)^{\frac{\kappa-1}{\kappa}} - 1\right] \tag{51}$$

Because $c^2 = \kappa\frac{p}{\rho}$ we get by (51)

$$\underline{w}^2 = c^2\,\mu = \frac{2\kappa}{\kappa - 1}\frac{p}{\rho}\left[\left(\frac{p_o}{p}\right)^{\frac{\kappa-1}{\kappa}} - 1\right]$$

hence

$$q = \frac{1}{2}\rho\,\underline{w}^2 = \frac{\kappa}{\kappa - 1}\,p\left[\left(\frac{p_o}{p}\right)^{\frac{\kappa-1}{\kappa}} - 1\right] \tag{52}$$

Let

$$f(\mu) = \frac{q}{p_o} = \frac{\kappa}{\kappa - 1}\frac{p}{p_o}\left[\left(\frac{p_o}{p}\right)^{\frac{\kappa-1}{\kappa}} - 1\right] \tag{53}$$

In this instance

$$\frac{p - p_o}{q_\infty} = \frac{1}{f(\mu_\infty)}\left[\left(1 + \frac{\kappa - 1}{2}\mu\right)^{-\frac{\kappa}{\kappa-1}} - 1\right]$$

$$\frac{p_\infty - p_o}{q_\infty} = \frac{1}{f(\mu_\infty)}\left[\left(1 + \frac{\kappa - 1}{2}\mu_\infty\right)^{-\frac{\kappa}{\kappa-1}} - 1\right]$$

that is finally

$$\frac{p - p_\infty}{\frac{\rho}{2} w_\infty^2} = \frac{1}{f(\mu_\infty)} \left(1 + \frac{\kappa - 1}{2} \mu\right)^{\frac{\kappa}{1-\kappa}} - \left(1 + \frac{\kappa - 1}{2} \mu_\infty\right)^{\frac{\kappa}{1-\kappa}} \tag{54}$$

the value from (44) to be inserted for μ.

X. EXAMPLE

The foregoing considerations are suggestive of exploring for the first those bodies of revolutions the meridian curve $\underline{C}$ of which is an analytical function of x in the (x, r) plane:

$$\underline{C} = r = r(x)$$

The example proceeds from the series of curves

$$r = d \frac{(n + 1)^{n+1}}{n^n} x^n (1 - x)$$

with

$$d > 0, \; 0 < n \leq 1, \; 0 \leq x \leq 1$$

The factor d gives a measure for the thickness of the body. If $0 < n < 1$ the profiles in the (x, r) plane have a round forebody and a pointed afterbody. For $n = 1$, $\underline{C}$ is symmetrical to $x = 0.5$, in which case the forebody and the afterbody are pointed.

Generally

$$|r|_{max} = d \quad \text{for} \quad x = \frac{n}{n + 1}$$

The values chosen for the parameters are

$$d = \frac{1}{20};\ n = \frac{2}{3};\ \lambda = 0.8\ (\text{hence}\ \mu_\infty = 0.64);\ \Lambda = +\left(1 - \lambda^2\right)^{-\frac{1}{2}} = \frac{5}{3}$$

Since $\bar{r} = \Lambda^{-1} r = 0.6\ r$, the meridian curve $\bar{C}$ of the affine transformed body lies within C. The meridian sections C and $\bar{C}$ are shown in figure 3.

The constant length a of the source distance was put at 0.1. After computing the coefficients (38), the system (37) is solved by iteration. Thereby it was found that the values z_i for the source intensities converge quicker than by the von Kármán method for the initial body in incompressible flow. Both distributions of vortex intensity for the same body of revolution in compressible and incompressible flow for the same flow velocity at infinity $U = 266.4\text{m/sec}^{-1}$, $\mu_\infty = 0.64$ are included in figure 3.

Figure 4 shows the velocity in terms of U; figure 5 the Mach number for compressible flow. The Mach number follows from (44) as function of the velocity $|w| = \frac{|v|}{u}\,|w_\infty|$, with $|w_\infty| = 0.3368$ according to figure 2 and equation (45).

Figure 6 represents the pressure distribution by (54) on the body along with the pressure distribution of incompressible flow for comparison.

The numerical values are given in table I.

XI. SUMMARY

The present report gives a method for determination of the velocity and pressure distributions as they appear on bodies of revolution in axially-symmetrical flow in the subsonic range of a gas with adiabatic equation of state.

The differential equation for the velocity potential Φ of the gas motion is first linearized in such a manner that in the coefficients of the second partial derivatives of Φ with respect to the coordinates for the velocity components u, v, w and the local sonic velocity c the corresponding values from infinity $U = u_\infty,\ 0,0;\ c_\infty$ are inserted. There results for Φ a linear partial differential equation of the second order of elliptic type with constant coefficients. By an affine transformation A of the variables it can be converted to the Laplace form $\bar{\Delta}\bar{\Phi} = 0$. The performed linearization is valid only outside of a certain neighborhood of the stagnation points.

By the transformation A the body of revolution with the meridian curve $\underline{C}: r = r(x)$ is transformed into a similar body normal to the axis of rotation with the meridian curve $\bar{\underline{C}}: \bar{r} = \bar{r}(x)$. $\bar{\Phi}$ as solution of the equation $\bar{\Delta}\bar{\Phi} = 0$ represents the velocity potential of an incompressible fictitious flow in a space $(\bar{y})$; it is to be noted that $\bar{\underline{C}}$ is not a streamline. On the contrary, the equipotential lines $\bar{\bar{\Phi}} =$ Constant intersect the curve $\bar{\underline{C}}$ not orthogonally. Thus $\bar{\Phi}$ is to be determined as a regular potential function which satisfies in the outer range of $\bar{C}$ the equation $\bar{\Delta}\bar{\Phi} = 0$, at infinity has the form $\bar{\Phi} = U\,\bar{x}$ + reg. potential function, and on $\bar{\underline{C}}$ fulfills the condition

$$\frac{1}{2}\frac{\partial\bar{\Phi}}{\partial\bar{x}}\frac{dr^2}{dx} - \bar{r}\frac{\partial\bar{\Phi}}{\partial\bar{r}} = 0$$

This last requirement results from conversion of the boundary condition $\partial\Phi/\partial n = 0$ to $\underline{C}$ for the velocity potential Φ.

As in the incompressible medium there exists for $\bar{\Phi}$ a function $\bar{\Psi}$ "pseudo-stream function" which satisfies the conditions

$$\frac{\partial\bar{\Psi}}{\partial\bar{x}} = -\bar{r}\frac{\partial\bar{\Phi}}{\partial\bar{r}};\ \frac{\partial\bar{\Psi}}{\partial\bar{r}} = \bar{r}\frac{\partial\bar{\Phi}}{\partial\bar{x}}$$

in the outer range of $\bar{\underline{C}}$ satisfies the equation

$$\frac{\partial^2\bar{\Psi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\Psi}}{\partial\bar{r}^2} - \frac{1}{r}\frac{\partial\bar{\Psi}}{\partial\bar{r}} = 0$$

at infinity has the form $\bar{\Psi} = \frac{1}{2}U\bar{r}^2$ + regular function, and on $\bar{\underline{C}}$ satisfies the equation:

$$\Lambda\frac{\partial\bar{\Psi}}{\partial\bar{r}}\frac{dr}{dx} + \frac{\partial\bar{\Psi}}{\partial\bar{x}} = 0 \qquad (55)$$

with

$$\Lambda = + \left(1 - \frac{\bar{\mu}_\infty}{c_\infty}\right)^{-\frac{1}{2}} = + (1 - \mu_\infty)^{-\frac{1}{2}}$$

Under the assumption that the function $\bar{\Phi}$ can be analytically continued into the interior of the body to the symmetry axis, $\bar{\Phi}$ can be produced by a constant superposition of sources on the axis. Von Kármán's method mentioned at the beginning is based on this assumption: the continuous source distribution on the axis is replaced by a constant superposition in sections. The number of source lengths is selected according to the desired degree of accuracy. Corresponding to this number n one chooses on the contour $\bar{C}$ n varying "control points;" most serviceably, on the center perpendiculars of the source lengths. The solution of the integral equation for the continuous source distribution is reduced to a linear inhomogeneous equation system of the nth order, the unknown source intensities determined so that they satisfy the equation (55) in the control points.

The solution of this equation system by iteration converges more rapidly for a body in compressible medium than for the same body in a fluid of constant volume, the reason is on the one hand the form of the coefficients which is more symmetrical than von Kármán's, on the other hand the greater slenderness of the auxiliary body ($\bar{C}$) which results by the transformation A.

The pressure distribution is determined from Bernoulli's equation and the equation of state. It is advantageous to introduce the limiting velocity $\bar{w}$ (outflow velocity of the gas into the vacuum). Thus it is possible to determine the Mach number at every point of the contour and to compare pressure distributions for various Mach numbers.

As example the pressure distributions for a certain body of revolution were calculated in the compressible medium ($\mu_\infty = 0.64$) and in the medium of constant volume. The results are compiled in table I.

Translated by J. Vanier
National Advisory Committee
for Aeronautics

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2. Kaplan, C.: Two-Dimensional Subsonic Compressible Flow Past Elliptic Cylinders. NACA Rep. No. 624, 1938.

TABLE I

i	x	$\lvert r \rvert$	$\frac{d\lvert r \rvert}{dx}$	z_i compressible	z_i incompressible	$\frac{\lvert v \rvert}{U}$ compressible	$\frac{\lvert v \rvert}{U}$ incompressible	$\frac{p - p_\infty}{q}$ compressible	$\frac{p - p_\infty}{q}$ incompressible	$+\sqrt{\mu}$
	0	0	∞			0	0	1.17049	1.0	0
1	.05	.01979	0.16729	0.03236	0.03760	0.98054	(0.98054)	.03904	.04091	.78249
2	.15	.03684	.12037	.04611	.06227	1.00808	1.02172	-.01601	-.04392	.80734
3	.25	.04569	.06091	.02909	.01574	1.02988	1.02795	-.05974	-.05668	.82712
4	.35	.04955	.02078	.01087	.02255	1.03439	1.03028	-.06964	-.06147	.83088
5	.45	.04958	-.01616	-.00874	-.02240	1.03170	1.02803	-.06430	-.05686	.82906
6	.55	.04637	-.04666	-.02274	-.01217	1.02417	1.02067	-.04820	-.04178	.82176
7	.65	.04031	-.07369	-.03061	-.04275	1.01342	1.01509	-.02425	-.03042	.81176
8	.75	.03168	-.09854	-.03169	-.02202	1.00111	99876	-.00193	-.00247	.80090
9	.85	.02066	-.12134	-.02476	-.03563	.98859	.99067	.02305	-.01858	.78973
10	.95	.00742	-.14314	-.01017	-.00103	.98734	.97174	.02591	-.05571	.78830
	1.0	0	$-\infty$			0	0	1.17049	1.0	0

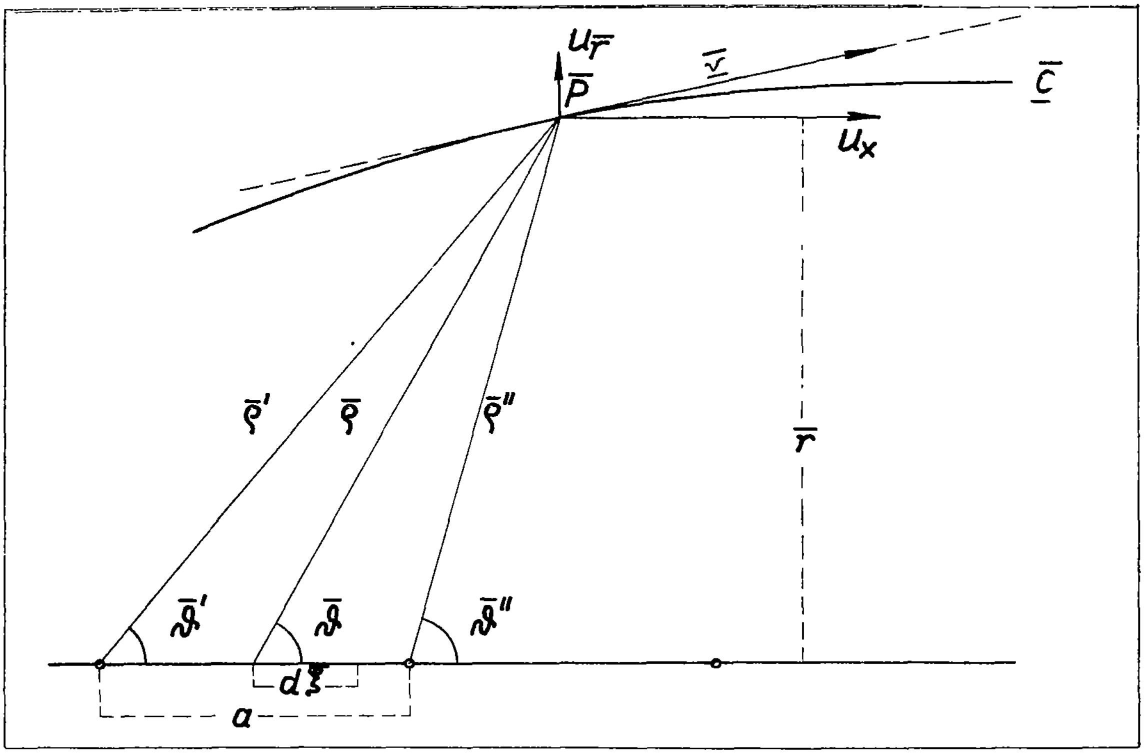

Figure 1.-

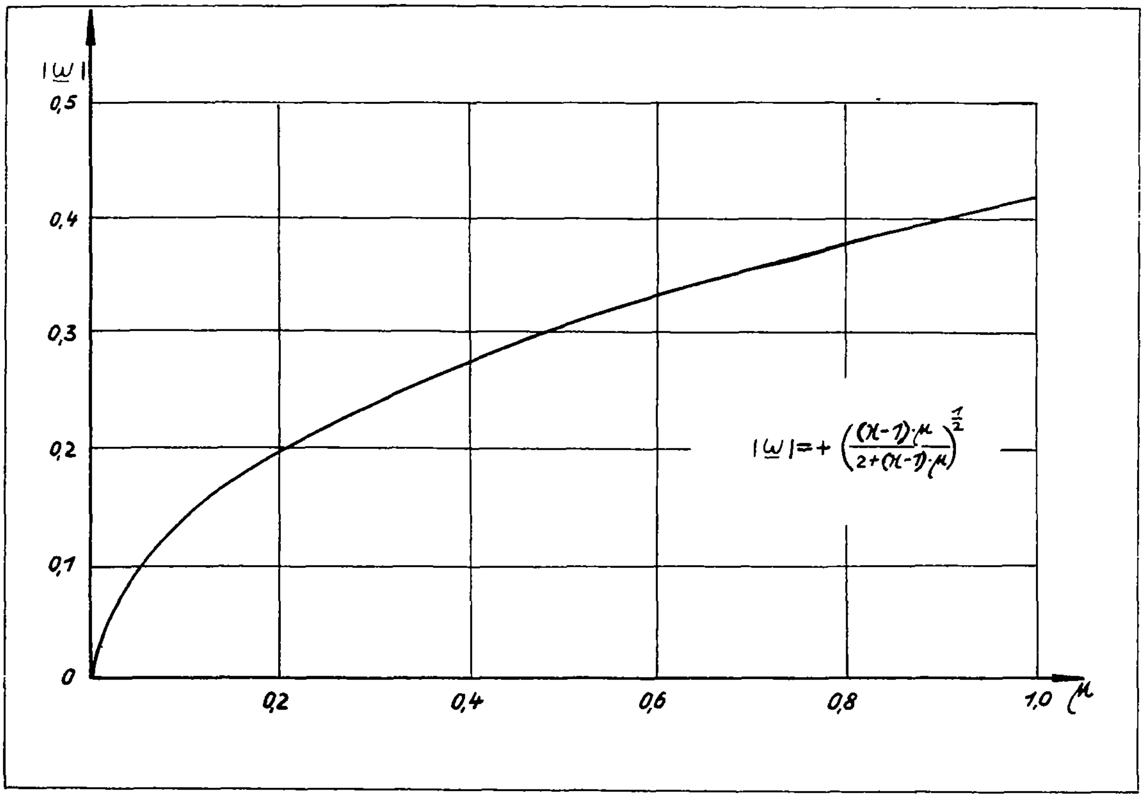

Figure 2.-

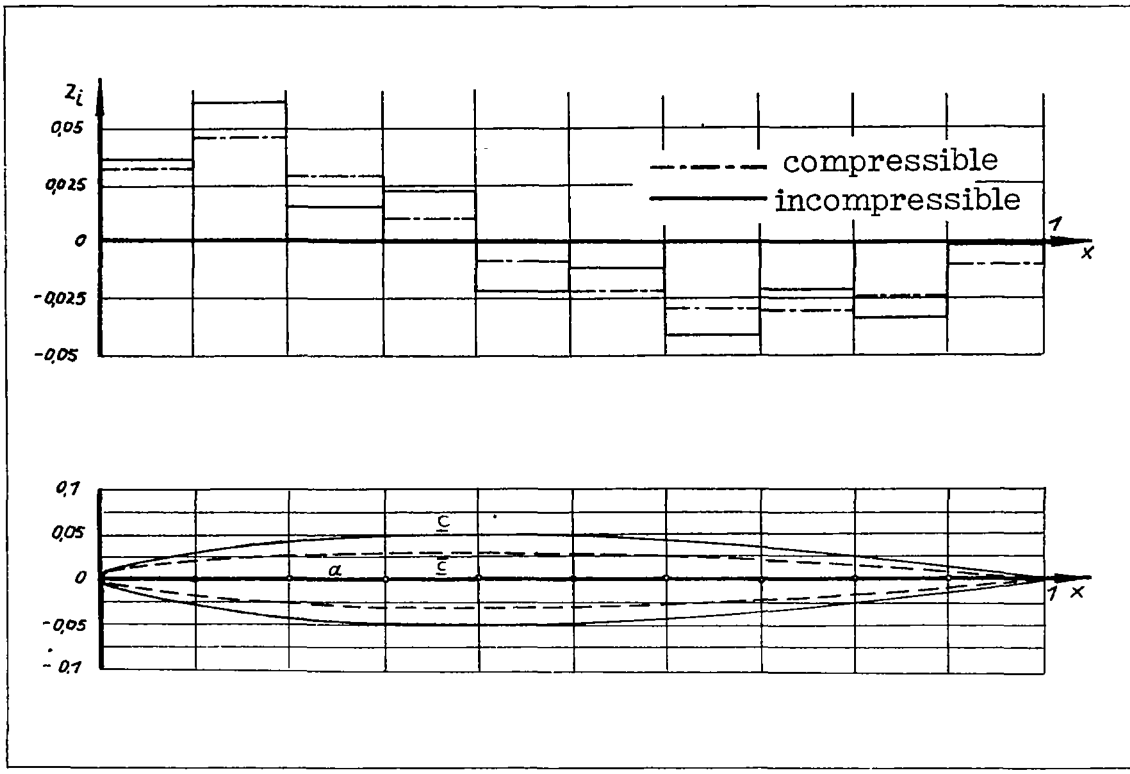

Figure 3.- Meridian Sections $\underline{C}$ and $\overline{\underline{C}}$; dimensionless source strengths z_i.

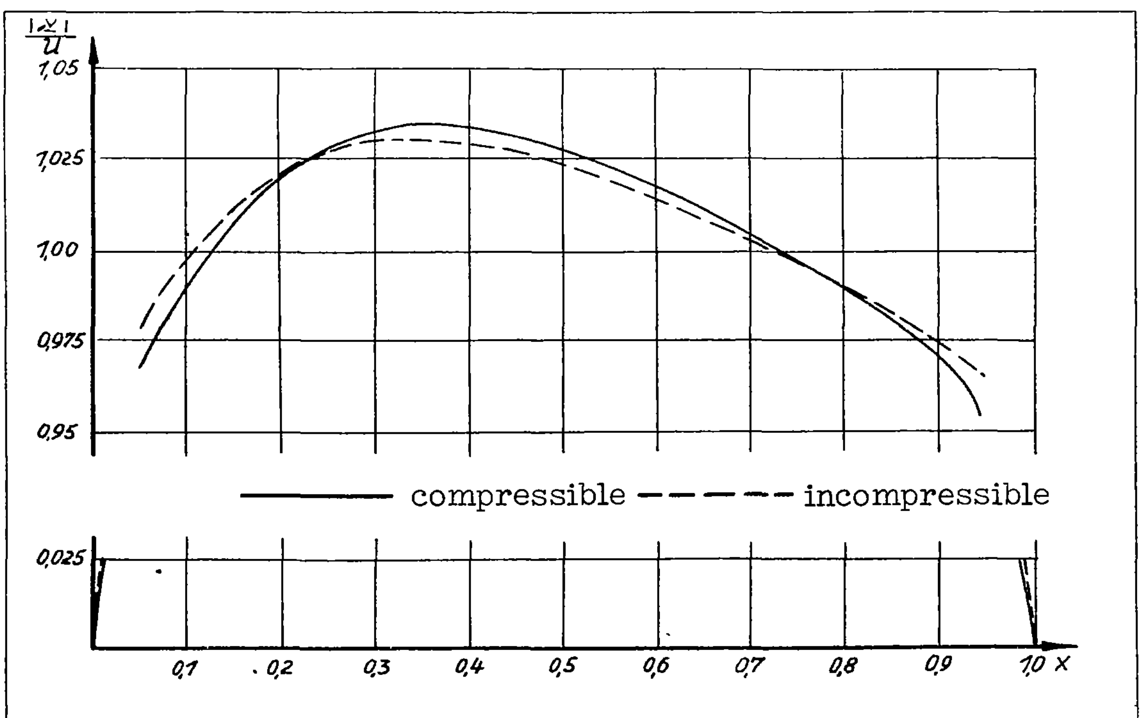

Figure 4.- Velocity distribution on the body of revolution along a meridian.

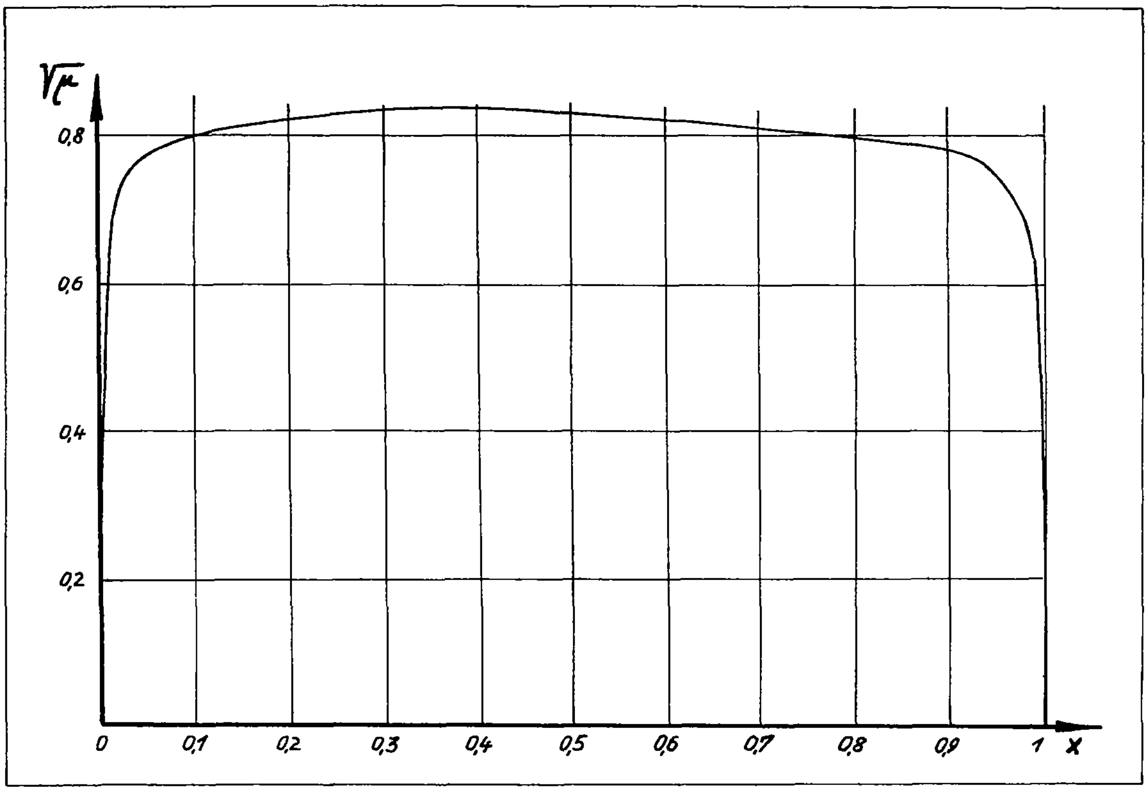

Figure 5.- Curve of the Mach number along a meridian.

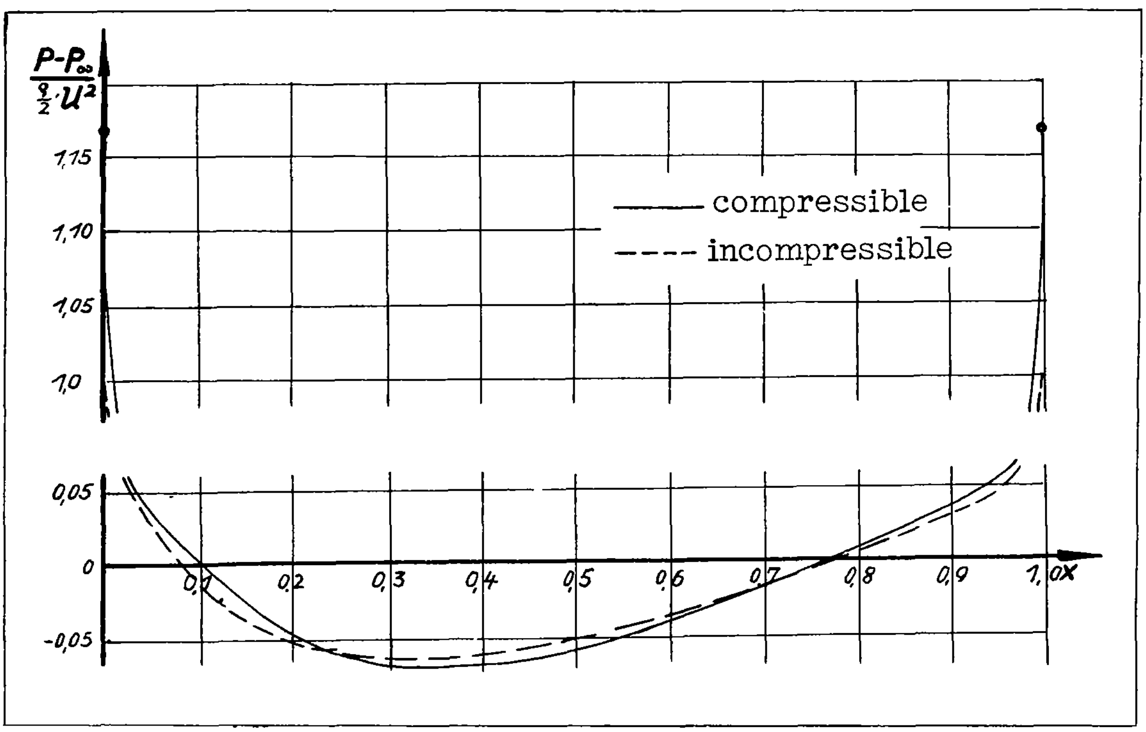

Figure 6.- Pressure distribution.